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Pinning of an Interface by a Weak Potential

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We prove that in a two-dimensional Gaussian SOS model with a small attractive potential the height of the interface remains bounded no matter how small the potential is; this is in sharp contrast with the free situation in which the interface height diverges logarithmically in the thermodynamic limit.

KEY WORDS: Interfaces; Solid-On-Solid model; cluster expansions.

1. INTRODUCTION. DEFINITION OF THE INTERFACE MODEL

A two-dimensional interface in a three-dimensional translation-invariant continuum has fluctuations which diverge as the logarithm of the size of the system in the thermodynamic limit. The same is true on lattice systems at temperatures higher than the roughening temperature and, of course, lower than the critical temperature. This effect can be understood in terms of capillary waves of unbounded wavelengths, or in terms of the infrared divergence of a two-dimensional massless field. It is remarkable, however, that an arbitrarily small perturbation favoring a particular localization of the interface is enough to make the fluctuations bounded around the preferred localization. The present paper gives a rigorous proof of this fact for a simple model where the interface is described by heights $h_i \in \mathbb{R}$ for $i \in \mathbb{Z}^2$, coupled through the Hamiltonian

$$H = \sum_{|i-j|=1} (h_i - h_j)^2 + \sum_i V(h_i) \tag{1.1}$$

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where the pinning potential $V(h)$ is like the one described in Fig. 1 or Fig. 2. The boundedness of the fluctuations for this model was obtained previously in mean field approximation.[2] The analogous problem for a one-dimensional interface in a two-dimensional space is equivalent to the existence of a bound state for a small attractive potential in one-dimensional quantum mechanics (ref. 1 and references therein). Since the infrared divergences are weaker in two dimensions than in one, our result is not surprising. However, it seems to us that neither the one-dimensional result nor the technique of its proof can be used in two dimensions. In two dimensions there are rigorous results on mass generation based on random walk correlation inequalities,[6] but they are restricted to models where the potential $V(h)$ in (1.1) is monotone, increasing at least logarithmically at infinity.

The idea of our proof is new and is based on a kind of Peierls argument, suitably scaled: the analog of a "+" spin is a height $|h_i| \leqslant a$, which feels the pinning potential; the other heights $|h_i| > a$ being regarded as "−" spins. We look at a state with "+" boundary conditions. The lattice $\mathbb{Z}^2$ is divided into square blocks of side $l \gg (a\varepsilon)^{-1/2} \gg 1$. For each configuration of the h_i a block is called a "+" block if at least one of the l^2 spins it contains is a "+" spin. It is called a "−" block if it contains only "−" spins. A contour is then a connected loop of "+" blocks encircling a region of "—" blocks. Therefore, inside the loop, the interface is not allowed to visit the potential and $e^{-V(\text{inside})} = 1$ for such a contour. This weight "1" can be compared to the weight of the other configurations for which the connected loop of "+" blocks is present, but where the inside of the loop is not specified. The probability that $|h_i| \leqslant a$ at any given site i inside is then at least what it would be in the absence of the potential inside, but still with the same conditioning along the loop. This probability can be estimated to be of order $2a/(\ln L)^{1/2}$ if L is the size (perimeter) of the loop. It follows that

$$\langle e^{-V(\text{inside})} \rangle \sim \exp\left(-\frac{2a}{(\ln L)^{1/2}}\, \varepsilon c \cdot L^2 \right) \tag{1.2}$$

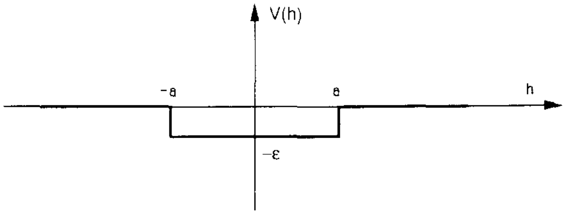

Fig. 1

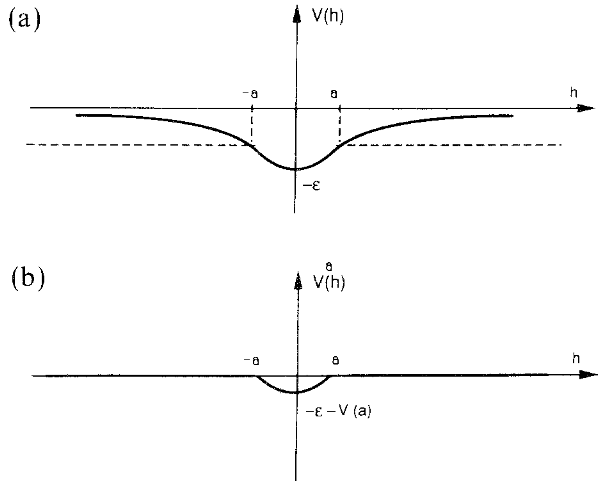

Fig. 2

where ε is the strength of the potential and $c \cdot L^2$ counts the number of sites inside the loop (for a typical loop like a square or a disk). Therefore the weight of a contour of "+" blocks encircling a region of "—" blocks should be bounded by

$$\exp\left(-\frac{2a}{(\ln L)^{1/2}}\,\varepsilon c \cdot L^2\right) \tag{1.3}$$

which becomes nicely small as the length L of the contour tends to $+\infty$. The probability of an isolated "—" block should similarly be bounded by

$$\exp\left(-\frac{2a}{(\ln l)^{1/2}}\,\varepsilon l^2\right) \tag{1.4}$$

which is small if

$$l/(\ln l)^{1/4} \gg (a\varepsilon)^{-1/2} \tag{1.5}$$

The final result of all this scaled Peierls expansion should then be

$$\langle |h_i| \rangle \sim (\ln l)^{1/2} \leqslant [\ln(a\varepsilon)^{-1}]^{1/2} \tag{1.6}$$

As usual, various technical difficulties come into the game. A notable one is "entropic repulsion": high interface regions (insides of contours) see a potential barrier between $h = -a$ and $h = a$; entropic repulsion[3] then pushes the interface away as far as $\ln l$ instead of the normal fluctuations $(\ln l)^{1/2}$. This is the main reason why we finally obtain only

$$\langle |h_i| \rangle \leqslant 3a + K|\ln \bar{a}\bar{\varepsilon}| \tag{1.7}$$

with $\bar{a} = \inf(a, 1)$ and $\bar{\varepsilon} = \inf(\varepsilon, 1/2)$ (the factor 3 is not optimal and one could replace it by any number bigger than one).

Our analysis does not apply yet to the decay of the correlation functions, hence to the computation of the "longitudinal correlation length" (the height of the interface being usually called the "transverse correlation length"). This is clearly an interesting open problem. We do not know whether it can be solved with correlation inequalities such as the ones used in this paper. If correlation inequalities fail, one might also try to attack this problem with a phase space expansion.

Let us now turn to precise definitions and results. Background information on solid-on-solid and other interface models can be found in ref. 1.

Let Λ be a finite volume in $\mathbb{Z}^2$, which is a large square for simplicity: $\Lambda = \mathbb{Z}^2 \cap [-L, L]^2$, and let $d\mu_0^\Lambda$ be the free Gaussian measure with Dirichlet boundary conditions, i.e.,

$$d\mu_0^\Lambda = \left\{ \exp\left[-\sum_{\langle i,j \rangle} (h_i - h_j)^2 \right] \right\} \prod_{i \in \Lambda} dh_i \prod_{i \in \Lambda^c} \delta_{h_i,0} \tag{1.8}$$

where the sum is over nearest-neighbor sites and Λ^c is the complement of Λ in $\mathbb{Z}^2$. In the rest of this paper expectation values such as $\langle \cdots \rangle_{dv}$ of an observable always refer to its mean value with respect to the measure dv; subscripts are used to remind the reader of the particular measure considered. It will be convenient to use the notation $\langle \cdots \rangle_0$ instead of $\langle \cdots \rangle_{d\mu_0^\Lambda}$.

By an easy Gaussian computation the mean value $\langle h_i^2 \rangle_0$ at any fixed site i diverges logarithmically as $\Lambda \to \infty$, i.e., as the thermodynamic limit is performed.

We add now a small interacting potential which tends to confine h_i in a neighborhood of 0. For convenience we choose

$$V(h) = 0 \qquad \text{if} \quad |h| > a \tag{1.9a}$$

$$V(h) = -\varepsilon \qquad \text{if} \quad |h| \leqslant a \tag{1.9b}$$

with a and ε both positive. We define the measure

$$d\mu_V^\Lambda = d\mu_0^\Lambda \prod_{i \in \Lambda} e^{-V(h_i)} \tag{1.10}$$

and we will use the notation $\langle\cdots\rangle_V$ for the expectation value with respect to the measure $d\mu_V^\Lambda$.

We shall also write for convenience $d\mu_V^\Lambda = d\mu_0^\Lambda e^{-V}$.

The rest of this paper is devoted to a proof that the expectation value of $|h_i|$ remains bounded as Λ goes to infinity and that the upper bound is a fixed function of a and of the integral $a\varepsilon$ of the potential. More precisely, we shall prove:

Theorem 1. There exists a constant K such that for any fixed $i \in \Lambda$

$$\sup_{\Lambda \to \infty} \langle |h_i| \rangle_V \leqslant 3a + K|\ln \bar{a}\bar{\varepsilon}| \tag{1.11}$$

where $\bar{a} = \inf(a, 1)$ and $\bar{\varepsilon} = \inf(\varepsilon, 1/2)$. The use of $\bar{\varepsilon}$ instead of ε is purely technical and simply prevents $|\ln \bar{a}\bar{\varepsilon}|$ from vanishing.

We are in fact insterested mostly in the regime where $a\varepsilon \ll 1$, where the bound (1.11) diverges logarithmically in the product $a\varepsilon$, which is the area of the potential. We must stress here that the bound (1.11) is not optimal; we could derive more precise bounds using the remark following Lemma 2.3, but they would be not as simple as (1.11).

The method we used can be easily generalized to the case of any negative even potential V increasing on $\mathbb{R}^+$. Let $a > 0$ and define the potential V_a by

$$V_a(h) = V(h) - V(a) \qquad \text{if} \quad |h| \leqslant a \tag{1.12a}$$

$$V_a(h) = 0 \qquad \text{if} \quad |h| > a \tag{1.12b}$$

(Fig. 2b); then we have the following result.

Theorem 2. There exists a constant K such that for any fixed $i \in \Lambda$

$$\sup_{\Lambda \to \infty} \langle |h_i| \rangle_V \leqslant 3a + K \left| \ln \left[- \int_{-a}^{a} V^a(h)\, dh \right] \right| \tag{1.13}$$

We can of course choose a such that it minimizes the right-hand side of (1.13). From

$$\frac{d}{da} \int_{-a}^{a} V^a(h)\, dh = -2a \frac{d}{da} V(a)$$

the optimal choice of a is obtained for

$$\frac{d}{da} V(a) = -\frac{3}{K} \frac{1}{2a} \int_{-a}^{a} V^a(h)\, dh \tag{1.14}$$

The proof of Theorem 2 is a simple extension of the proof of Theorem 1, and is given in Appendix D. Therefore we imagine from now on that we are in the case of the potential of Fig. 1 and turn to the proof of Theorem 1.

2. PROOF OF THEOREM 1

2.1. High and Low Interface Decomposition

To study the model, we put it first in a big finite box Λ, then we prove estimates which are uniform in Λ and let $\Lambda \to \infty$ (thermodynamic limit). Although this is not essential, we assume that the total volume Λ that we consider is the union of a large number of blocks of side l. The corresponding finite set of blocks is noted D_0^Λ.

Let us repeat that for any given block $\Delta \in D_0^\Lambda$ there are two possibilities: either all the interface in Δ has absolute value above a and Δ is a high interface block, or at some site $j \in \Delta$ we have $|h_j| \leqslant a$ and Δ is a low interface block. We define Γ as the set of blocks of D_0^Λ with high interface, i.e., for which the first possibility holds, and we call it the high interface region (its complement being obviously called the low interface region). We fix $i \in \Lambda$ and write

$$\begin{aligned}\langle |h_i| \rangle_V &= \frac{\sum_\Gamma \langle |h_i|\, \chi_\Gamma \rangle_V}{\sum_\Gamma \langle \chi_\Gamma \rangle_V} \\ &= \frac{\sum_{\Gamma \not\ni \Delta_0} \langle |h_i|\, \chi_\Gamma \rangle_V}{\sum_\Gamma \langle \chi_\Gamma \rangle_V} + \frac{\sum_{\Gamma \ni \Delta_0} \langle |h_i|\, \chi_\Gamma \rangle_V}{\sum_\Gamma \langle \chi_\Gamma \rangle_V}\end{aligned} \tag{2.1}$$

where χ_Γ is the characteristic function of the event that the high interface region is exactly Γ, i.e.,

$$\chi_\Gamma((h_i)_{i \in \Lambda}) = \prod_{\Delta \in \Gamma} \chi(\forall i \in \Delta,\, |h_i| > a) \prod_{\Delta \notin \Gamma} \chi(\exists i \in \Delta,\, |h_i| \leqslant a) \tag{2.2}$$

and Δ_0 is the particular block of D_0^Λ to which the site i belongs. We have decomposed the sum according to whether in the numerator Δ_0 belongs to Γ or not, and we bound the corresponding terms by different arguments.

2.2. Upper Bound When *i* is In the Low Interface Region (i.e., $\Delta_0 \notin \Gamma$)

Let us define ψ_Δ as the characteristic function of the event that Δ has low interface, i.e.,

$$\psi_\Delta((h_i)_{i \in \Delta}) = \chi(\exists i \in \Delta,\, |h_i| \leqslant a) \tag{2.3}$$

We have

$$\sum_{\Gamma \not\ni \Delta_0} \chi_\Gamma = \psi_{\Delta_0} \tag{2.4}$$

hence

$$\frac{\sum_{\Gamma \not\ni \Delta_0} \langle |h_i| \chi_\Gamma \rangle_V}{\sum_\Gamma \langle \chi_\Gamma \rangle_V} = \frac{\langle |h_i| \psi_{\Delta_0} \rangle_V}{\langle 1 \rangle_V} = \langle |h_i| \psi_{\Delta_0} \rangle_V \tag{2.5}$$

For α subset of Λ let ϕ_α be the characteristic function of the event that α is exactly the set of sites j in Λ for which $|h_j| \leqslant a$, i.e.,

$$\phi_\alpha((h_i)_{i \in \Lambda}) = \prod_{i \in \alpha} \chi(|h_i| \leqslant a) \prod_{i \notin \alpha} \chi(|h_i| > a) \tag{2.6}$$

We have

$$\psi_{\Delta_0} = \sum_{\alpha \cap \Delta_0 \neq \varnothing} \phi_\alpha \tag{2.7}$$

hence

$$\langle |h_i| \psi_{\Delta_0} \rangle_V = \sum_{\alpha \cap \Delta_0 \neq \varnothing} \langle |h_i| \phi_\alpha \rangle_V \tag{2.8}$$

By definition we denote the expectation value with respect to $\phi_\alpha \, d\mu_V$ or $\phi_\alpha \, d\mu_0$ as $\langle \cdots \rangle_{\alpha, V}$ or $\langle \cdots \rangle_{\alpha, 0}$. Hence

$$\langle |h_i| \rangle_{\alpha, V} = \frac{\langle |h_i| \phi_\alpha \rangle_V}{\langle \phi_\alpha \rangle_V} \tag{2.9}$$

Using the GKS inequality (Appendix A), we obtain

$$\langle |h_i| \rangle_{\alpha, V} \leqslant \langle |h_i| \rangle_{\alpha, 0} \tag{2.10}$$

Indeed, $\phi_\alpha \, d\mu_V$ is an even ferromagnetic measure, and $\exp(V)$ is a positive element of $\mathscr{C}$ (Appendix A); hence, as a result

$$\langle |h_i| \rangle_{\alpha, 0} = \langle |h_i| \rangle_{\phi_\alpha \, d\mu_0} = \langle |h_i| \rangle_{\exp(V) \phi_\alpha \, d\mu_V} \geqslant \langle |h_i| \rangle_{\phi_\alpha \, d\mu_V} \tag{2.11}$$

By the GKS inequality we have again

$$\langle |h_i| \rangle_{\alpha, 0} \leqslant \langle h_i \rangle_{h_j = a, \forall j \in \alpha; h_j \geqslant a, \forall j \notin \alpha} \tag{2.12}$$

where the subscript 0, which indicates that expectations are computed with respect to $d\mu_0$, not $d\mu_V$, is omitted from now on for simplicity. We have to be careful at this point, because the event $(h_j = a,\ \forall j \in \alpha;\ h_j \geqslant a,\ \forall j \notin \alpha)$ is of measure 0. What is strictly defined is

$$\langle h_i \rangle_{h_j \in [a-\delta, a], \forall j \in \alpha; h_j \geqslant a, \forall j \notin \alpha}$$

As a function of δ this expression has a finite limit when δ goes to 0. Inequality (2.12) holds true because if we define the function

$$\begin{aligned} \rho(\{h\}) &\equiv \prod_{j \in \alpha} \chi(h_j > 0)\, \chi(|h_j| \geqslant a - \delta) \prod_{j \notin \alpha} \chi(h_j > 0)\, \chi(|h_j| > a) \\ &= \prod_{j \in \alpha} \frac{1+\sigma_j}{2} \chi(|h_j| \geqslant a - \delta) \prod_{j \notin \alpha} \frac{1+\sigma_j}{2} \chi(|h_j| > a) \end{aligned} \tag{2.13}$$

(where σ_j is the sign of h_j as defined in Appendix A), then ρ is in the positive cone $\mathscr{C}$ defined in Appendix A. When multiplied by ϕ_α this function becomes the characteristic function of the event $(h_j \in [a-\delta, a],\ \forall j \in \alpha;\ h_j \geqslant a,\ \forall j \notin \alpha)$. Using the GKS inequality (see Appendix A)

$$\langle |h_i| \rangle_{\rho\phi_\alpha} \geqslant \langle |h_i| \rangle_{\alpha, 0} \tag{2.14}$$

When δ goes to 0, we obtain the announced inequality (2.12).

Let j_0 be one particular site of α; once again from the GKS inequality we have

$$\langle h_i \rangle_{(h_j = a, \forall j \in \alpha; h_j \geqslant a, \forall j \notin \alpha)} \leqslant \langle h_i \rangle_{(h_{j_0} = a, h_j \geqslant a, \forall j \neq j_0)}$$

The left-hand side of (2.12) is equal to

$$a + \langle h_i \rangle_{h_{j_0} = 0, h_j \geqslant 0, \forall j \neq j_0}$$

It remains therefore to find an upper bound to $\langle h_i \rangle_{h_{j_0} = 0, h_j \geqslant 0, \forall j \neq j_0}$. We are now faced with a SOS model above a wall. When the condition $(h_j \geqslant 0)$ is relaxed, $\langle |h_i| \rangle_{h_{j_0} = 0}$ is bounded by $C[\ln \bar{d}(i, j_0)]^{1/2}$, with $\bar{d} = d + 2$, the Euclidean distance plus 2 (the factor 2 is added to ensure that $\ln \bar{d}$ is always strictly positive, and C is some constant).

The condition $h_j \geqslant 0$, $\forall j$, can be considered as a "wall" which generates an entropy repulsion phenomenon,[3] and changes the behavior of this one-point function. Namely we have the following result.

Lemma 2.1. There exists two constants C_1, C_1' such that

$$\langle h_i \rangle_{h_{j_0} = 0, h_j \geqslant 0, \forall j \neq j_0} \geqslant C_1' \ln \bar{d}(i, j_0) \tag{2.15a}$$

$$\langle h_i \rangle_{h_{j_0} = 0, h_j \geqslant 0, \forall j \neq j_0} \leqslant C_1 \ln \bar{d}(i, j_0) \tag{2.15b}$$

The first inequality (2.15a) is proved in ref. 3. The proof of the second inequality (2.15b) (the only one we use in this paper) is contained in Appendix B.

Assuming Lemma 2.1, we obtain

$$\langle |h_i| \rangle_{\alpha, V} \leqslant a + C_1 \ln \bar{d}(i, j_0) \tag{2.16}$$

Using $\bar{d}(i, j_0) \leqslant 2 + l\sqrt{2}$ and $\sum_\alpha \langle \phi_\alpha \rangle_V \leqslant 1$, we can return to (2.5), (2.8), and (2.9) and conclude that

$$\frac{\sum_{\Gamma \not\ni \Delta_0} \langle |h_i| \, \chi_\Gamma \rangle_V}{\sum_\Gamma \langle \chi_\Gamma \rangle_V} \leqslant a + C_1 \ln(2 + l\sqrt{2}) \tag{2.17}$$

In the next sections we have to choose l proportional to $(\bar{a}\varepsilon)^{-1/2}$ (see Lemma 2.3 below). With this choice the left-hand side of (2.17) is bounded by the right-hand side of (1.11), as desired.

2.3. Upper Bound When i is in the High Interface Region (i.e., $\Delta_0 \in \Gamma$)

We consider that two blocks Δ and Δ' of D_0 are connected if they share a common edge (a common corner is not enough). When Δ_0 has high interface the high interface region Γ can be decomposed into connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$, where by definition Γ_0 is the connected component of Γ containing Δ_0. The set of connected components of Γ is called $C(\Gamma)$.

We define also the boundary $\partial\Gamma$ of a set Γ of blocks as made of that set of blocks not in Γ, but sharing an edge with some block of Γ (Fig. 3).

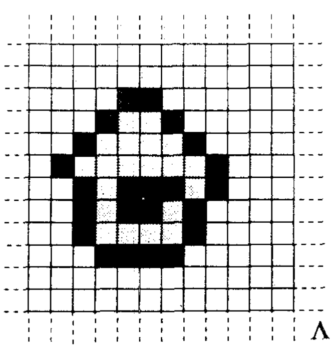

Fig. 3

We have

$$\left(\sum_{\Gamma \ni \Delta_0} \langle |h_i|\, \chi_\Gamma \rangle_V\right) \Big/ \left(\sum_{\Gamma} \langle \chi_\Gamma \rangle_V\right)$$

$$= \left(\sum_{\substack{\Gamma_0 \ni \Delta_0 \\ \Gamma_0 \text{ connected}}} \sum_{\Gamma, C(\Gamma) \ni \Gamma_0} \langle |h_i|\, \chi_\Gamma \rangle_V \right) \Big/ \left(\sum_{\Gamma} \langle \chi_\Gamma \rangle_V\right)$$

$$= \sum_{\substack{\Gamma_0 \ni \Delta_0 \\ \Gamma_0 \text{ connected}}} \left[\left(\sum_{\Gamma, C(\Gamma) \ni \Gamma_0} \langle |h_i| \rangle_{\Gamma, V} \langle \chi_\Gamma \rangle_V \right) \Big/ \left(\sum_{\Gamma} \langle \chi_\Gamma \rangle_V\right) \right] \tag{2.18}$$

where again $\langle \cdots \rangle_{\Gamma, V}$ is a natural notation for the expectations with respect to the measure $\chi_\Gamma \, d\mu_V$. We are following once more the method of the last section to bound $\langle |h_i| \rangle_{\Gamma, V}$ in a way which depends only on the distance from i to $\partial \Gamma_0$.

Let $A(\Gamma)$ be the following set:

$$A(\Gamma) = \{\alpha \subset \Lambda, \forall \Delta \in \Gamma, \alpha \cap \Delta = \varnothing, \forall \Delta \notin \Gamma, \alpha \cap \Delta \neq \varnothing\} \tag{2.9}$$

so that $\chi_\Gamma = \sum_{\alpha \in A(\Gamma)} \phi_\alpha$. We write

$$\langle |h_i| \rangle_{\Gamma, V} = \frac{\sum_{\alpha \in A(\Gamma)} \langle |h_i|\, \phi_\alpha \rangle_V}{\sum_{\alpha \in A(\Gamma)} \langle \phi_\alpha \rangle_V} \tag{2.20}$$

As a result,

$$\langle |h_i| \rangle_{\Gamma, V} \leqslant \sup_{\alpha \in A(\Gamma)} \langle |h_i| \rangle_{\alpha, V} \tag{2.21}$$

From the GKS inequality,

$$\langle |h_i| \rangle_{\alpha, V} \leqslant \langle |h_i| \rangle_{\alpha, 0} \tag{2.22}$$

Let us choose a block Δ in $\partial \Gamma_0$ and j_0 an element of $\alpha \cap \Delta$. We can then use again inequality (2.12) to deduce

$$\langle |h_i| \rangle_{\alpha, 0} \leqslant \langle h_i \rangle_{h_{j_0} = a, h_j \geqslant a, \forall j \neq j_0} \tag{2.23}$$

Let $q(i, \partial \Gamma_0)$ be the smallest number of elementary blocks connecting i to $\partial \Gamma_0$. This is a natural lattice definition of the distance from i to the boundary of Γ_0. From (2.16) we obtain

$$\langle |h_i| \rangle_{\alpha, V} \leqslant a + C_1 \ln L(i, \partial \Gamma_0), \qquad \forall \alpha \in A(\Gamma) \tag{2.24}$$

with

$$L(i, \partial \Gamma_0) = lq(i, \partial \Gamma_0) + 2l\sqrt{2} + 2 \tag{2.25}$$

Therefore

$$\sum_{\substack{\Gamma_0 \text{ connected} \\ \Delta_0 \in \Gamma_0}} \frac{\sum_{\Gamma, C(\Gamma) \ni \Gamma_0} \langle |h_i| \rangle_{\Gamma, V} \langle \chi_\Gamma \rangle_V}{\sum_\Gamma \langle \chi_\Gamma \rangle_V}$$

$$\leqslant \sum_{\substack{\Gamma_0 \text{ connected} \\ \Delta_0 \in \Gamma_0}} p(\Gamma_0)[a + C_1 \ln L(i, \partial \Gamma_0)] \tag{2.26}$$

where $p(\Gamma_0)$ is the probability that in the theory with potential V the high interface region Γ has Γ_0 as one of its connected components. In mathematical notation

$$p(\Gamma_0) = \frac{\sum_{\Gamma, C(\Gamma) \ni \Gamma_0} \langle \chi_\Gamma \rangle_V}{\sum_\Gamma \langle \chi_\Gamma \rangle_V} \tag{2.27}$$

The rest of our proof consists in showing that $p(\Gamma_0)$ decreases sufficiently fast with the size of Γ_0 so as to allow the summation over Γ_0 in (2.26) and to offset the large factor $[a + C_1 \ln L(i, \partial \Gamma_0)]$.

This is done in two steps.

Lemma 2.2. These exists a constant C_2 such that

$$\ln p(\Gamma_0) \leqslant - \sum_{\Delta \in \Gamma_0} \frac{C_2 a \varepsilon l^2}{a + \ln L(\Delta, \partial \Gamma_0)} \tag{2.28}$$

where $L(\Delta, \partial \Gamma_0)$ is defined by (2.25) applied at the center of the block Δ.

Lemma 2.3. There exists a constant $C_6 > 0$ such that for $\bar{a} \varepsilon l^2 / (\ln l) > C_6$

$$\sum_{\substack{\Gamma_0 \text{ connected} \\ \Delta_0 \in \Gamma_0}} [a + C_1 \ln L(i, \partial \Gamma_0)] \exp\left(- \sum_{\Delta \in \Gamma_0} \frac{C_2 a \varepsilon l^2}{a + \ln L(\Delta, \partial \Gamma_0)} \right) \leqslant 2(a+1) \tag{2.29}$$

Combining Lemmas 2.2 and 2.3 with (2.18)–(2.26) and with the low interface bound (2.17) completes the proof of (2.11), hence of the theorem.[2]

Now we turn to the proof of these two technical lemmas.

[2] We have in fact a better upper bound than (2.29). Indeed from the proof of Lemma (2.3) it can be shown that there exists a constant M such that for $\bar{a}\varepsilon l^2/(\ln l)$ large enough, the upper bound in (2.29) can be chosen equal to $2(a+1)\exp[-M\bar{a}\varepsilon l^2/(\ln l)]$.

2.4. Proof of Lemma 2.2

First of all, if Γ_0 is in $C(\Gamma)$, then $\partial\Gamma_0$ is in the low interface region; as a result

$$\begin{aligned}\sum_{\Gamma, C(\Gamma) \ni \Gamma_0} \langle \chi_\Gamma e^{-V_\Lambda} \rangle_0 &= \sum_{\Gamma, C(\Gamma) \ni \Gamma_0} \langle \chi_\Gamma e^{-V_{\Lambda - \Gamma_0}} \rangle_0 \\ &\leqslant \sum_\Gamma \langle \chi_\Gamma \psi_{\partial\Gamma_0} e^{-V_{\Lambda-\Gamma_0}} \rangle_0 \\ &\leqslant \langle \psi_{\partial\Gamma_0} e^{-V_{\Lambda-\Gamma_0}} \rangle_0 \end{aligned} \tag{2.30}$$

with $\psi_{\partial\Gamma_0} = \prod_{\Delta \in \partial\Gamma_0} \psi_\Delta$, ψ_Δ being defined in (2.3).

We also have

$$\begin{aligned}\sum_\Gamma \langle \chi_\Gamma e^{-V_\Lambda} \rangle_0 &\geqslant \sum_\Gamma \langle \chi_\Gamma \psi_{\partial\Gamma_0} e^{-V_\Lambda} \rangle_0 \\ &= \langle \psi_{\partial\Gamma_0} e^{-V_\Lambda} \rangle_0 \end{aligned} \tag{2.31}$$

As a result we obtain the bound

$$p(\Gamma_0) \leqslant \frac{\langle \psi_{\partial\Gamma_0} e^{-V_{\Lambda-\Gamma_0}} \rangle_0}{\langle \psi_{\partial\Gamma_0} e^{-V_\Lambda} \rangle_0} = (\langle e^{-V_{\Gamma_0}} \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}})^{-1} \tag{2.32}$$

Using Jensen's inequality, it follows that

$$p(\Gamma_0) \leqslant \exp(\langle V_{\Gamma_0} \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}}) \tag{2.33}$$

By definition of V we have

$$\langle V_{\Gamma_0} \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}} = -\varepsilon \sum_{i \in \Gamma_0} \langle \chi(|h_i| \leqslant a) \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}} \tag{2.34}$$

It remains to find a lower bound on

$$\langle \chi(|h_i| \leqslant a) \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}}$$

This is accomplished by the following lemma:

Lemma 2.4. We have, if C_1 is the constant appearing in (2.15b),

$$\langle \chi(|h_i| \leqslant a) \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}} \geqslant \frac{a}{4a + 4C_1 \ln L(i, \partial\Gamma_0)} \tag{2.35}$$

From this bound and (2.33)–(2.34) we immediately get Lemma 2.2.

The proof of Lemma 2.4 is divided into three steps:

1. The following inequality holds true:

$$\langle |h_i| \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}} \leqslant a + C_1 \ln L(i, \partial\Gamma_0) \tag{2.36}$$

2. The function

$$x \mapsto \langle \delta(h_i - x) \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}}$$

is an even function, which is decreasing on $\mathbb{R}^+$.

3. Conclusion: proof of Lemma 2.4.

Proof of 1.

$$\begin{aligned} \langle |h_i| \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}} &= \frac{\sum_{\alpha \in A(\partial\Gamma_0)} \langle |h_i| \phi_\alpha \rangle_{V_{\Lambda-\Gamma_0}}}{\sum_{\alpha \in A(\partial\Gamma_0)} \langle \phi_\alpha \rangle_{V-\Gamma_0}} \\ &\leqslant \sup_{\alpha \in A(\partial\Gamma_0)} \langle |h_i| \rangle_{\alpha, V_{\Lambda-\Gamma_0}} \end{aligned} \tag{2.37}$$

Using (2.16), this is less than $a + C_1 \ln L(i, \partial\Gamma_0)$.

Proof of 2. This is the most delicate part. Let us denote by F the function which assigns to each $x \in \mathbb{R}$ the real number

$$F(x) = \langle \delta(h_i - x) \rangle_{\psi_{\partial\Gamma_0}, V_{\Lambda-\Gamma_0}}$$

We have

$$F(x) = \frac{\langle \delta(h_i - x) \psi_{\partial\Gamma_0} \exp(-V_{\Lambda-\Gamma_0}) \rangle}{\langle \psi_{\partial\Gamma_0} \exp(-V_{\Lambda-\Gamma_0}) \rangle} \tag{2.38}$$

Therefore it is necessary and sufficient to show that the function

$$x \mapsto G(x) = \langle \delta(h_i - x) \psi_{\partial\Gamma_0} \exp(-V_{\Lambda-\Gamma_0}) \rangle$$

is even (which is obvious) and decreasing on $\mathbb{R}^+$.

From translation invariance

$$\begin{aligned} &\langle \delta(h_i - x) \psi_{\partial\Gamma_0} \exp(-V_{\Lambda-\Gamma_0}) \rangle \\ &= \Big\langle \delta(h_i) \prod_{\Delta \in \partial\Gamma_0} \chi(\exists j \in \Delta, h_j \in [-a-x, a-x]) \\ &\quad \times \exp\Big\{ \sum_{j \in \Lambda-\Gamma_0} \varepsilon \cdot \chi(h_j \in [-a-x, a-x]) \Big\} \Big\rangle \end{aligned} \tag{2.39}$$

As a result

$$\frac{d}{dx}G(x)=\left\langle \delta(h_i)\frac{d}{dx}\left(\prod_{\Delta\in\partial\Gamma_0}\chi(\exists j\in\Delta,\, h_j\in[-a-x,\,a-x])\right.\right.$$
$$\left.\left.\times\exp\left\{\sum_{j\in\Lambda-\Gamma_0}\varepsilon\cdot\chi(h_j\in[-a-x,\,a-x])\right\}\right)\right\rangle$$
$$=\sum_{\Delta\in\partial\Gamma_0}\left\langle\delta(h_i)\prod_{\substack{\Delta'\in\partial\Gamma_0\\ \Delta'\neq\Delta}}\chi(\exists j\in\Delta',\, h_j\in[-a-x,\,a-x])\right.$$
$$\times\frac{d}{dx}\{\chi(\exists j\in\Delta,\, h_j\in[-a-x,\,a-x])\}$$
$$\left.\times\exp\left\{\sum_{j\in\Lambda-\Gamma_0}\varepsilon\cdot\chi(h_j\in[-a-x,\,a-x])\right\}\right\rangle$$
$$+\sum_{j\in\Lambda-\Gamma_0}\left\langle\delta(h_i)\prod_{\Delta\in\partial\Gamma_0}\chi(\exists j\in\Delta,\, h_j\in[-a-x,\,a-x])\right.$$
$$\times\frac{d}{dx}(\varepsilon\cdot\chi(h_j\in[-a-x,\,a-x]))$$
$$\left.\times\exp\left\{\sum_{j\in\Lambda-\Gamma_0}\varepsilon\cdot\chi(h_j\in[-a-x,\,a-x])\right\}\right\rangle \tag{2.40}$$

We shall show that each individual term in this sum is negative. First of all we have

$$\frac{d}{dx}\chi(\exists j\in\Delta,\, h_j\in[-a-x,\,a-x])$$
$$=\sum_{j\in\Delta}\frac{d}{dx}(\chi(h_j\in[-a-x,\,a-x]))\prod_{j'\neq j}\chi(h_{j'}\notin[-a-x,\,a-x]) \tag{2.41}$$

This can easily be proved by induction on the number of points of Δ.

Moreover, using the identity

$$\frac{d}{dx}\chi(h\in[-a-x,\,a-x])=[\delta(h+a+x)-\delta(h-a+x)] \tag{2.42}$$

we finally obtain, using once more translation invariance,

$$-\frac{d}{dx}G(x)=\sum_{\Delta\in\partial\Gamma_0,\,j\in\Delta}\left\langle\delta(h_i-x)[\delta(h_j-a)-\delta(h_j+a)]\right.$$
$$\left.\times\prod_{\substack{j'\in\Delta\\ j'\neq j}}\chi(h_{j'}\notin[-a,a])\prod_{\substack{\Delta'\in\partial\Gamma_0\\ \Delta'\neq\Delta}}\psi_{\Delta'}e^{-V_{\Lambda-\Gamma_0}}\right\rangle$$
$$+\varepsilon\sum_{j\in\Lambda-\Gamma_0}\left\langle\delta(h_i-x)[\delta(h_j-a)-\delta(h_j+a)]\prod_{\Delta\in\partial\Gamma_0}\psi_\Delta e^{-V_{\Lambda-\Gamma_0}}\right\rangle \tag{2.43}$$

We have

$$\left\langle\delta(h_i-x)[\delta(h_j-a)-\delta(h_j+a)]\prod_{\substack{j'\in\Delta\\ j'\neq j}}\chi(h_{j'}\notin[-a,a])\prod_{\substack{\Delta'\in\partial\Gamma_0\\ \Delta'\neq\Delta}}\psi_{\Delta'}e^{-V_{\Lambda-\Gamma_0}}\right\rangle$$
$$=\frac{1}{2}\left\langle[\delta(h_i-x)-\delta(h_i+x)][\delta(h_j-a)-\delta(h_j+a)]\right.$$
$$\left.\times\prod_{\substack{j'\in\Delta\\ j'\neq j}}\chi(h_{j'}\notin[-a,a])\prod_{\substack{\Delta'\in\partial\Gamma_0\\ \Delta'\neq\Delta}}\psi_{\Delta'}e^{-V_{\Lambda-\Gamma_0}}\right\rangle \tag{2.44}$$

Applying (2.44), the positivity of each element of the sum (2.43) is then a straightforward consequence of the following fact: let $F(h_1,..., h_N)$ be a positive function on $\mathbb{R}^N$ which is an even function of each variable. Let $h_i=\sigma_i|h_i|$; then

$$\left\langle\prod_{k=1}^{p}\sigma_{i_k}F(h_1,...,h_N)\right\rangle_{d\mu_0}\geqslant 0 \tag{2.45}$$

The proof of this result is obtained by expanding the off-diagonal part of $d\mu_0$ and by integrating over the "spins" $\{\sigma_i\}$, using the obvious inequalities

$$\langle\sigma_{i_1}\cdots\sigma_{i_p}\rangle\geqslant 0$$

Proof of 3. Since

$$F(x)=\langle\delta(h_i-x)\rangle_{\psi_{\partial\Gamma_0},\,V_{\Lambda-\Gamma_0}}$$

the function F is positive, even, and we have $\int_0^{+\infty}F(x)\,dx=1/2$. Let $z\geqslant 0$ be such that $\int_0^z F(x)\,dx=\frac{1}{4}$, and let

$$X=2\int_0^{+\infty}xF(x)\,dx=\langle|h_i|\rangle_{\psi_{\partial\Gamma_0},\,V_{\Lambda-\Gamma_0}}$$

From

$$z\int_{z}^{+\infty} F(x)\,dx \leqslant \int_{z}^{+\infty} xF(x)\,dx \leqslant X/2 \tag{2.46}$$

we deduce

$$\frac{1}{z} \geqslant \frac{1}{2X} \tag{2.47}$$

Because F is monotone decreasing on $\mathbb{R}^+$ its mean value on an interval $[0, x]$ is also decreasing, hence the following inequality holds:

$$\frac{1}{a}\int_{0}^{a} F(x)\,dx \geqslant \frac{1}{z}\int_{0}^{z} F(x)\,dx \tag{2.48}$$

if $z \geqslant a$. Therefore,

$$\int_{0}^{a} F(x)\,dx \geqslant \frac{a}{4z} \geqslant \frac{a}{8X} \tag{2.49}$$

Using (2.36), we end up with

$$\int_{0}^{a} F(x)\,dx \geqslant \frac{a}{8a + 8C_1 \ln L(i, \partial\Gamma_0)} \tag{2.50}$$

This is the announced lower bound (2.35); hence, it completes the proof of Lemma 2.4 if $z \geqslant a$.

If $z \leqslant a$, then

$$\int_{0}^{a} F(x)\,dx \geqslant \int_{0}^{z} F(x)\,dx = \frac{1}{4} \geqslant \frac{a}{8a + 8C_1 \ln L(i, \partial\Gamma_0)} \tag{2.51}$$

From Lemma 2.4 and (2.33)–(2.34), Lemma 2.2 follows.

2.5. Proof of Lemma 2.3. Let us start with some informal discussion on this lemma before turning to the technical proof.

Lemma 2.3 is nontrivial because usually in high-temperature cluster expansions or situations of this type one has exponential decay in the volume (here the area) of the clusters or polymers on which summation has to be performed (the dilute case). Here we do not have exponential decay because it is spoiled by the logarithm of the distance to the boundary; for a typical round, convex cluster of radius r we have therefore only decay exponential in $r^2/\ln r$.

Near the boundary of the cluster the troublesome logarithmic factor is, however, bounded, so that we have clearly exponential decay in the length of the boundary of the cluster. Our first observation is that for convex clusters or in fact for simply connected ones in two dimensions, knowing its boundary is enough to determine the cluster. Hence decay in the size of the boundary is in fact sufficient for summing over simply connected objects. In two dimensions we can imagine any connected cluster as a simply connected cluster with a finite set of holes punched in it. Hence the problem becomes a summation over the positions and shapes of the holes. The shape of the holes is controlled by the decay in the size of their boundaries (there are no holes in the holes in this problem). Still there remains a subtlety: one has also to sum over the positions of the holes. Clearly this is a problem only when the number of holes gets large. The solution must exploit the fact that when in a simply connected region there are many holes randomly distributed, then each point in the region is not far from a hole. Therefore in this case the logarithmic distance which spoiled the exponential decay in the area of the region is typically not big so that in average it does not really spoil anything. This point of view is nice, but to exploit it rigorously we have to order the holes in some systematic way, grouping together the ones which are closest to one another, then grouping together the clusters of holes so obtained and iterating the process. To organize the summation in this way is possible and described in Appendix C; here we give another way to proceed, which is to decompose the space according to a series of lattices of increasing size. A convenient decomposition is the same decomposition with a geometric progression of sizes that us natural for renormalization group or phase space analysis (see, e.g., ref. 4 for a review), although the argument will in fact be much simpler here.

We consider fixed lattices $D_1,..., D_k,...$ which generalize D_0. The lattice D_k is made of squares of side size $l \cdot 2^k$ and each square Δ of D_k contains exactly four squares of D_{k-1}. We choose such a dyadic decomposition for simplicity, but of course other rules would be possible. In fact, we work with the restricted lattices D_k^Λ made of the squares in D_k which are entirely within Λ, so that for fixed Λ we have only a finite set of finite lattices to consider, but our estimates are uniform in Λ and the superscript Λ will be omitted most of the time for simplicity.

The region Γ_0 that we are interested in is called Γ from now on for simplicity and it is decomposed according to the sequence of lattices D_k in the following way.

Any square in D_k which is entirely contained inside Γ is said to belong to $I_k(\Gamma)$, the interior of Γ at scale k. In particular, Γ itself can be identified with $I_0(\Gamma)$. Since Γ is fixed and finite, there is an index L such that for $k > L$, $I_k(\Gamma)$ is empty. A square Δ in $I_k(\Gamma)$ such that the square Δ' in D_{k+1}

containing Δ does not belong to $I_{k+1}(\Gamma)$ is called a maximal square of Γ. The corresponding set of such maximal squares at scale k is called $M_k(\Gamma)$, and the set of all maximal squares in Γ is $M(\Gamma) = \bigcup_{k=0}^{L} M_k(\Gamma)$. Remark that Γ is exactly equal to the disjoint union of its maximal squares: $\Gamma = \bigcup_{\Delta \in M(\Gamma)} \Delta$. In particular, since in this section Δ_0 belongs to Γ, there is a unique scale called k_0 such that $\Delta_0 \in M_{k_0}(\Gamma)$. The particular square of $M_{k_0}(\Gamma)$ containing Δ_0 is called $\Delta_0^{k_0}$.

Each square of D_0 in a maximal square of $M_k(\Gamma)$ is at a distance at most $2\sqrt{2}\, 2^k$ from the border of Γ, and there are 4^k such squares. Therefore there exist constants C_3, C_4, and C_5 such that

$$\sum_{\substack{\Gamma \text{ connected} \\ \Delta_0 \in \Gamma}} \exp - \left(\sum_{\Delta \in \Gamma} \frac{C_2 a \varepsilon l^2}{a + \ln L(\Delta, \partial\Gamma)} \right) [a + C_1 \ln L(i, \partial\Gamma)]$$
$$\leqslant \sum_{\substack{\Gamma \text{ connected} \\ \Delta_0 \in \Gamma}} \exp - \left(\sum_k C_3 a \varepsilon l^2 \, |M_k(\Gamma)| \frac{4^k}{a + k + \ln l} \right) [a + C_4 k_0]$$
$$\leqslant \sum_{k_0} f(a, \varepsilon, l, k_0) \sum_{\substack{\Gamma \text{ connected} \\ \Delta_0 \subset M_{k_0}(\Gamma)}} \exp \left(- \sum_k C_5 \bar{a} \varepsilon l^2 \, |M_k(\Gamma)| \frac{3^k}{\ln l} \right) \tag{2.52}$$

with

$$f(a, \varepsilon, l, k_0) \equiv [a + C_4(k_0 + \ln l)] \exp \left(-C_5 \bar{a} \varepsilon l^2 \frac{3^{k_0}}{\ln l} \right) \tag{2.53}$$

and $\bar{a} = \inf(a, 1)$. At given values of a and ε, $\sum_{k_0} f(a, \varepsilon, l, k_0)$ can be made as small as we want by choosing l large enough. [In (2.52) we bounded $a4^k/(a + \ln l + k)$ by const $\cdot\, 3^k$ $(\bar{a}/\ln l)$, which of course is not optimal.]

When $M(\Gamma)$ is reduced to $M_0(\Gamma)$ [i.e., $M_k(\Gamma)$ is empty for $k > 0$] summing over Γ is a kind of "Koenigsberg bridge" problem completely standard in this context. One uses the decay factors $e^{-K|\Gamma|}$ with K large coming in this case from (2.52) (plus the fact that $\Delta_0 \in \Gamma$ to break translation invariance). This is usually done by associating to each connected set a different spanning tree and then a different random walk of length twice the area of the set obtained by "turning around the tree." The number of such random walks on a finite-dimensional lattice being exponential in their length, one ends up with a geometrically convergent series.

The only difference here is that the set Γ lives on various scales. Nevertheless, we follow the same strategy as in the standard case. Squares Δ and Δ' belonging to $M(\Gamma)$ (even within different lattices D_k and $D_{k'}$) are still said to be connected if they share a portion of an edge (a common corner is not enough). Recall that we consider Γ as the union of the squares in $M(\Gamma) \equiv \bigcup_{k=0}^{L} M_k(\Gamma)$. We draw the graph of the elementary relations of connectedness between the elements of $M(\Gamma)$. The fact that Γ itself is connected means that this graph is connected. Therefore, by eventually omitting some of the relations, for each such graph we can choose a particular spanning tree (in an arbitrary way). This tree is made of a labelled tree T with $n \equiv |M(\Gamma)|$ vertices labelled as 0, 1,..., $n-1$, and $n-1$ lines (in the usual mathematical sense of Cayley's theorem) together with an assignment ω of a square $\omega(i)$ of a certain scale k_i to each vertex i, $0 \leqslant i \leqslant n-1$, of the tree. The set of all squares $\omega(i)$ is then nothing but $M(\Gamma)$.

We can always choose ω such that the vertex with label 0 (the root of the tree) is $\Delta_0^{k_0}$ in $M_{k_0}(\Gamma)$ [in other words, $\omega(0) = \Delta_0^{k_0}$].

In this way, knowing k_0, the sum over the sets Γ can be bounded by the sum over the trees T and the assignments ω such that the root of the tree is at $\Delta_0^{k_0}$, and such that squares assigned by ω to vertices which are linked on the tree are connected. Remark, however, that in this way we associate truly $(n-1)!$ pairs (T, ω) (with the features specified above) to each set Γ because of the permutational symmetry for the labelling of the vertices of T. It is important to take this overcounting factor into account.

Another important observation is that for a fixed $\Delta \in M_k$ there are $4 \cdot 2^{k-k'}$ squares Δ' in $M_{k'}$ with $k' \leqslant k$ connected to it and 0, 1, or 2 squares Δ' in $M_{k'}$ connected to it for any $k' > k$. Hence, the maximal number of squares of a given scale k' connected to Δ is always bounded by $4 \cdot 2^k$.

Therefore (k_0 being fixed) we can bound the sum over Γ in (2.52) in the following way:

$$\sum_{\substack{\Gamma \text{ connected} \\ \Delta_0 \subset M_{k_0}(\Gamma)}} \exp -\left(\sum_k C_5 \frac{\bar{a}\varepsilon l^2}{\ln l} |M_k(\Gamma)|\, 3^k \right)$$
$$\leqslant \sum_{T,\omega} \frac{1}{(n-1)!} \exp -\left(\sum_{i=0}^{n-1} C_5 \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i} \right) \tag{2.54}$$

where we recall that k_i is the scale of the square $\omega(i)$.

By Cayley's theorem the number of trees T with coordination number d_i at site i is

$$d_i = \frac{(n-2)!}{\prod_i (d_i - 1)!} \tag{2.55}$$

(d_i, the coordination number or number of links in the tree ending at vertex i, is always bigger than or equal to 1). Therefore we have [since $(n-2)!/(n-1)! \leqslant 1$]

$$\sum_{T,\omega} \frac{1}{(n-1)!} \exp\left(-\sum_{i=0}^{n-1} C_5 \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right)$$
$$\leqslant \sum_{n\geqslant 1} \sum_{\{k_i\}} \sum_{\{d_i\}} \frac{1}{\prod_i (d_i-1)!} \sup_{\substack{T \text{ with} \\ n,\,\{d_i\} \text{ fixed}}} \sum_{\substack{\omega \text{ with} \\ \{k_i\} \text{ fixed}}} \exp -\left(\sum_{i=0}^{n-1} C_5 \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right) \tag{2.56}$$

To perform the sum over ω with the set of integers k_i fixed, we have simply to start from the known root $\omega(0)$ and to proceed toward the "end of the branches" of the trees. For fixed values of the $\{k_i\}$, fixed T, and fixed coordination numbers d_i, the sum over ω is bounded by $\prod_{i=0}^{n-1} [4\cdot 2^{k_i}]^{d_i-1}$, using the remark above on the number of squares of a given scale connected to a given one.

Therefore

$$\sum_{n\geqslant 1} \sum_{\{k_i\}} \sum_{\{d_i\}} \frac{1}{\prod_i (d_i-1)!} \sup_{\substack{T \text{ with} \\ n,\,\{d_i\} \text{ fixed}}} \sum_{\substack{\omega \text{ with} \\ \{k_i\} \text{ fixed}}} \exp\left(-\sum_{i=0}^{n-1} C_5 \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right)$$
$$\leqslant \sum_{n\geqslant 1} \sum_{\{k_i\}} \exp\left(-\sum_{i=0}^{n-1} \frac{C_5}{2} \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right)$$
$$\times \sum_{\{d_i\}} \prod_{i=0}^{n-1} \left[\frac{[4\cdot 2^{k_i}]^{d_i-1}}{(d_i-1)!} \exp\left(-\frac{C_5}{2} \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right)\right]$$
$$\leqslant \sum_{n\geqslant 1} \sum_{\{k_i\}} \exp -\left(\sum_{i=0}^{n-1} \frac{C_5}{2} \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right) \prod_{i=0}^{n-1} \left[\exp\left(4\cdot 2^{k_i} - \frac{C_5}{2} \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right)\right] \tag{2.57}$$

Choosing l large enough, so that

$$\frac{C_5}{2} \frac{\bar{a}\varepsilon l^2}{\ln l} \geqslant C_6 \geqslant \sup(4, C_4) \tag{2.58}$$

a condition that we assume now, we obtain

$$\sum_{n\geqslant 1} \sum_{\{k_i\}} \exp\left(-\sum_{i=0}^{n-1} \frac{C_5}{2} \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right) \prod_{i=0}^{n-1} \left[\exp\left(4\cdot 2^{k_i} - \frac{C_5}{2} \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_i}\right)\right]$$
$$\leqslant \sum_{n\geqslant 1} \sum_{\{k_i\}} \exp\left(-\sum_{i=0}^{n-1} 4C_6 3^{k_i}\right)$$
$$\leqslant \sum_n \left(\frac{e^{-4C_6}}{1-e^{-4C_6}}\right)^n = \frac{e^{-4C_6}}{1-2e^{-4C_6}} \leqslant 2e^{-4C_6} \tag{2.59}$$

Taking into account (2.52) and (2.53), our final bound, assuming that (2.58) is fulfilled, is simply

$$\begin{aligned} &2[\exp(-4C_6)] \sum_{k_0} f(a, \varepsilon, l, k_0) \\ &\quad = 2[\exp(-4C_6)] \sum_{k_0} [a + C_4 k_0] \exp\left(-C_5 \frac{\bar{a}\varepsilon l^2}{\ln l} 3^{k_0}\right) \\ &\quad \leqslant 2(a+1)\exp(-4C_6) \end{aligned} \tag{2.60}$$

[using (2.58)]. This completes the proof of Lemma 2.3.

APPENDIX A

In this appendix we recall the second GKS inequality, which we used throughout this paper. A measure $d\mu$ is said to be an even ferromagnetic measure on $\mathbb{R}^n$ if there exist n even measures on $\mathbb{R}$, $dv_1,..., dv_n$, and a matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ satisfying $a_{ij} \leqslant 0$ for $i \neq j$, such that

$$d\mu(x_1,..., x_n) = \exp\left(-\sum_{i,j} a_{ij} x_i x_j\right) \prod_i dv_i(x_i) \tag{A1}$$

A function on $\mathbb{R}$ can be uniquely written as the sum of an even and odd function. As a result we have

$$\mathscr{F}(\mathbb{R}, \mathbb{R}) = \mathscr{F}^+ \oplus \mathscr{F}^- \tag{A2}$$

where $\mathscr{F}^+$ (resp. $\mathscr{F}^-$) is the vector space of odd (resp. even) functions.

Let $\mathscr{C}^+$ (resp. $\mathscr{C}^-$) be the subset of $\mathscr{F}^+$ (resp. $\mathscr{F}^-$) consisting of increasing and positive functions on $\mathbb{R}^+$. Let $\mathscr{C}$ be the set $\mathscr{C}^+\mathscr{C}^-$. The set $\mathscr{C}$ is invariant under positive linear combinations and under multiplication.

Let σ be the function assigning to each x its sign [i.e., $\sigma(x) = x/|x|$ if $x \neq 0$, $\sigma(0) = 0$].

It is easily shown that $\mathscr{C}$ is the smallest set containing σ, $\mathscr{C}^+$, and products and sums of these sets. We can now construct a set $\mathscr{G}$ of functions on $\mathbb{R}^n$ (which has a positive cone structure) defined as follows: each element g of $\mathscr{G}$ can be written

$$g = \sum_i g_i \tag{A3}$$

where g_i has the following expression:

$$g_i(h_1,..., h_n) = \prod_j f_{ij}(h_j) \qquad \text{and} \qquad f_{ij} \in \mathscr{C} \tag{A4}$$

We can now state the second GKS inequality (a proof can be found in ref. 5). Let $d\mu$ be an even, positive ferromagnetic measure, and F and G be two elements of $\mathscr{G}$; then

$$\langle FG \rangle_{d\mu} \geqslant \langle F \rangle_{d\mu} \langle G \rangle_{d\mu} \tag{A5}$$

In particular, if G is positive, we obtain immediately

$$\langle F \rangle_{G d\mu} \geqslant \langle F \rangle_{d\mu} \tag{A6}$$

which is the inequality we used throughout this work.

APPENDIX B

In this appendix, we will derive an upper bound for the mean height of a fluctuating interface above a wall. We will precisely show inequality (2.15b), which we recall:

$$\langle h_i \rangle_{d\mu_0,\, h_j = 0,\, h_k \geqslant 0, \forall k} \leqslant C_1 \ln \bar{d}(i, j) \tag{B1}$$

Let χ_w be the characteristic function of the presence of the wall, i.e., $\chi_w(\{h_k\}) = \chi(h_k \geqslant 0,\ \forall k \in \Lambda)$. We have

$$\langle h_i \rangle_{d\mu_0,\, h_j = 0,\, h_k \geqslant 0, \forall k} = \frac{\langle h_i \chi_w \rangle_{h_j = 0}}{\langle \chi_w \rangle_{h_j = 0}} \tag{B2}$$

From the GKS inequality,

$$\langle h_i \rangle_{d\mu_0,\, h_j = 0,\, h_k \geqslant 0, \forall k \in \Lambda} \leqslant \langle h_i \rangle_{d\mu_0',\, h_j = 0,\, h_k \geqslant 0, \forall k \in \Lambda} \tag{B3}$$

where $d\mu_0'$ is the Gaussian measure with the following boundary condition:

$$d\mu_0' = \left\{ \exp\left[- \sum_{\langle k, k' \rangle} (h_k - h_{k'})^2 \right] \right\} \prod_{k \in \Lambda} dh_k \prod_{l \in \partial\Lambda} \delta_{h_l, f(l)} \tag{B4}$$

for any strictly positive function f on the boundary of Λ, the finite box in which the theory is defined. Of course, we are interested in bounds which are uniform as $\Lambda \to \infty$. The function f is chosen below for this purpose.

Furthermore, we can also introduce a strictly positive boundary condition $f(j)$ at site j which by GKS increases the value of $\langle h_i \rangle$. More precisely,

$$\begin{aligned} \langle h_i \rangle_{d\mu_0',\, h_j = 0,\, h_k \geqslant 0, \forall k} &\leqslant \langle h_i \rangle_{d\mu_0',\, h_j = f(j),\, h_k \geqslant 0, \forall k} \\ &= \langle h_i \chi_w \rangle_{d\mu_0',\, h_j = f(j)} / \langle \chi_w \rangle_{d\mu_0',\, h_j = f(j)} \\ &\leqslant \langle h_i \rangle_{d\mu_0',\, h_j = f(j)} / \langle \chi_w \rangle_{d\mu_0',\, h_j = f(j)} \end{aligned} \tag{B5}$$

We can write

$$1-\langle \chi_w \rangle_{d\mu_0', h_j=f(j)} = \langle \chi(\exists k \in \Lambda, h_k \leqslant 0) \rangle_{d\mu_0', h_j=f(j)}$$
$$\leqslant \sum_{k\in\Lambda} \langle \chi(h_k \leqslant 0) \rangle_{d\mu_0', h_j=f(j)} \tag{B6}$$

We shall now prove that for C large enough and for some suitable choice of f, when Λ goes to $+\infty$ we have the inequalities

$$\langle h_i \rangle_{d\mu_0', h_j=f(j)} \leqslant C' \ln \bar{d}(i, j) \tag{B7}$$

$$\sum_{k\in\Lambda} \langle \chi(h_k \leqslant 0) \rangle_{d\mu_0', h_j=f(j)} \leqslant \frac{1}{2} \tag{B8}$$

from which we easily deduce the upper bound (B1) with $C_1 = 2C'$. As a result the problem is reduced to the study of the one-point function:

$$\langle \chi(h_k = x) \rangle_{d\mu_0', h_j=f(j)} \tag{B9}$$

The random variable h_k being Gaussian, it is enough to know its mean value and its covariance to know entirely its distribution. The mean value $g(k)$ is the solution of a variational problem. The variational equations are

$$\Delta_d g(k) = 0 \qquad \forall k \in \Lambda, \quad k \neq (0, 0) \tag{B10}$$

$$g(j) = f(j); \qquad g(l) = f(l) \quad \forall l \in \partial\Lambda \tag{B11}$$

with Δ_d the discrete Laplacian on $\mathbb{Z}^2$, and f to be chosen below.

We begin by studying a similar problem in the continuum, where we choose a boundary condition which is $C \ln \bar{d}(i, j)$ on the boundary of a disk of radius 1 centered around the site j. The variational problem is

$$\Delta_c u(z) = 0 \qquad \forall z \in \mathbb{R}^2, \quad 1 < |z - j| \tag{B12}$$

$$u(z) = C \ln \bar{d}(i, j) \qquad \text{if} \quad |z - j| = 1 \tag{B13}$$

with Δ_c the continuous Laplacian. An obvious solution in polar coordinates ($r = |z - j|$) is, for any given positive constant C',

$$u(r) = C \ln \bar{d}(i, j) + C' \ln r \tag{B14}$$

We consider the discretized function u_d which is equal to u on $\mathbb{Z}^2 - \{(0, 0)\}$, and which is equal to $C \ln \bar{d}(i, j)$ at j. Finally, we define the function $v \equiv \Delta_d u_d$. This is a function on the lattice, and by explicit computation from the explicit form (B14) there exists a constant $K(C')$ (depending only on C') such that [with $r(k) = |k_j|$]

$$|v(k)| \leqslant K(C')/[r(k)+1]^3 \qquad \text{if} \quad |k| \in \Lambda, \quad k \neq j \tag{B15}$$

because for $k \neq j$, $\Delta_c u = 0$, and $\Delta_d u_d = (\Delta_d - \Delta_c)(u)$, which is evaluated by a third-order derivative [the constant piece $C \ln \bar{d}(i, j)$ disappears in the Laplacian].

By (B15), the integral I_v of v over the whole lattice $\mathbb{Z}^2$ is convergent and bounded by $K'(C')$. We consider on the lattice the function $w \equiv v - I_v \delta(k)$, i.e., the function w coincides with v except at the origin and its Fourier transform $\hat{w}$ vanishes at $p = 0$. From (B15) and the definition of w we have that, for some constant $K''(C')$,

$$|p|^{-1/2} |\hat{w}(p)| \leqslant K''(C') \tag{B16}$$

Since $|p|^{1/2}(4 - 2\cos p_1 - 2\cos p_2)^{-1}$ (which is the Fourier transform of the inverse of the discrete Laplacian) in integrable over $[-\pi, \pi]^2$, we obtain a uniform bound, which depends only on C' for the function e on $\mathbb{Z}^2$ whose Fourier transform is

$$\hat{e}(p) = \frac{1}{4 - 2\cos p_1 - 2\cos p_2} \hat{w}(p)$$

Therefore, if we consider the function $g \equiv u - e$, we have proved

$$|g(k) - C \ln \bar{d}(i, j) - C' \ln |k - j|| \leqslant L(C') \tag{B17}$$

for some positive function $L(C')$ depending only on C'.

Finally we take for the boundary condition of our discrete variational problem $f(l) = g(l)$ for $l \in \partial\Lambda$, and $f(j) = g(j)$. We solve the variational problem (B10)–(B11) with this boundary condition f (the solution exists and is unique because it corresponds to the diagonalization of a finite-dimensional quadratic form). But the solution of this variational problem is precisely g because the boundary condition is satisfied and $\Delta_d g = 0$ in $\Lambda - \{(0, 0)\}$.

The covariance of $h(k)$, when we effectuate the translation around the mean value $g(k)$, is the same as the covariance for the problem with 0 boundary conditions at site j and on the boundary of Λ. This covariance grows as $\Lambda \to \infty$, and its value in the thermodynamic limit is bounded in the standard way by $c \cdot \ln \bar{d}(j, k)$ for some constant c.

With this information, (B7)–(B8) follow easily. Indeed, (B7) is a simple consequence of (B17), and (B8) follows from

$$\langle \delta(h_k - x) \rangle_{d\mu_0', h_j = f(j)} \leqslant \exp\{-[x - g(k)]^2/[c \ln \bar{d}(j, k)]\} \tag{B18}$$

Using (B18), we can write

$$\begin{aligned}
&\sum_{k\in\Lambda} \langle \chi(h_k \leqslant 0)\rangle_{d\mu_0', h_j = f(j)} \\
&\qquad \leqslant \sum_{k\in\Lambda} \int_{-\infty}^{0} dx \exp \frac{-[x-g(k)]^2}{c\ln \bar{d}(j,k)} \\
&\qquad \leqslant \sum_{k\in\Lambda} \frac{c\ln \bar{d}(j,k)}{2g(k)} \exp \frac{-g(k)^2}{c\ln \bar{d}(j,k)} \\
&\qquad \leqslant \sum_{k\in\Lambda} \frac{c\ln \bar{d}(j,k)}{2g(k)} \exp \frac{-[C\ln \bar{d}(i,j) + C'\ln|k-j| - L(C')]^2}{c\ln \bar{d}(j,k)} \leqslant 1/2
\end{aligned} \tag{B19}$$

if we take first C' big enough, then C still much larger (depending on C').

APPENDIX C

In this appendix we explain how one can sum over the regions Γ, i.e., we give a second proof of Lemma 2.3, without using multiscale lattices. The region Γ is decomposed as $\Gamma' - \bigcup_{i=1}^{n} H_i$, where Γ' and each H_i are simply connected, hence it is entirely defined by its boundary (Fig. 4). The sum over all possible Γ's decomposes into a sum over Γ' and over all H_i. Let us describe the necessary steps to perform the summations in this way.

The small factors per cube in (2.29) allow one to sum over the shape of the simply connected regions Γ', H_i if we know one point of their boundary. Indeed the boundary of Γ' or H_i determines these regions, and

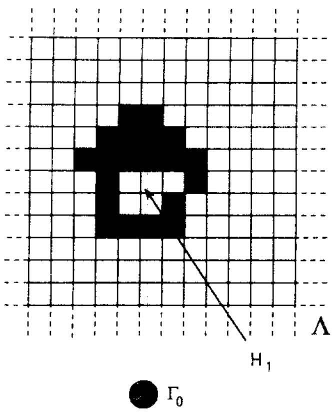

Fig. 4

on this boundary the bad factor $\ln(\Delta, \partial\Gamma)$ remains constant. Therefore we have an ordinary situation in which we have exponential decay in the size of the boundaries $\partial\Gamma'$ or ∂H_i, which allows us to sum over these boundaries if we know one particular cube of each of them.

The second fact is that we can sum the position of each hole one with respect to the other. This is done by building a tree connecting all these holes with some special properties. The rule to build this tree is the following. We draw the shortest among all the paths connecting two holes (if there are several such shortest paths, we choose one of them arbitrarily). This path connects two holes which are considered a single one from now on. Then we iterate this process, until the full tree is built (this recalls a cluster expansion à la Brydges–Battle–Federbuch; see, e.g., ref. 4). The tree built in this way has the following nice properties:

1. The lines in the tree do not cross.
2. Each square does not touch more than six lines in the tree.

Lines in the tree do not cross; otherwise, using the triangular inequality, one would find a contradiction with the fact that at each stage of the tree-building process the line added is the shortest between connected clusters already built.

The second property is true because six is the maximum number of tree lines ending in a given square because the hexagonal packing is the closest packing of spheres in two dimensions. For the same reason other squares are also not crossed by more than six lines.

These properties mean that giving one-sixth of the small factor of each square to each line which may cross it, we obtain an exponential decay in the length of each line. The endpoints of the lines are the particular squares on the boundaries of the holes which had to be fixed to sum over the boundaries. Organizing the summation in this way, we can prove Lemma 2.3. To perform the last summation (over Γ'), we have also to attach the boundary of Γ' to the particular square Δ_0 whose position is known and which is inside Γ. This is done by drawing one more line, the shortest one connecting Δ_0 to $\partial\Gamma$.

APPENDIX D

In this appendix we will generalize the method used in this paper to deal with a potential V negative, even, and increasing on $\mathbb{R}^+$. Let us define the function $V^a(h)$ by (1.12a)–(1.12b). V^a is negative, even, and increasing on $\mathbb{R}^+$. From the GKS inequality we have

$$\langle |h_i| \rangle_V \leqslant \langle |h_i| \rangle_{V^a} \tag{D1}$$

We again use the same arguments. Section 2.2 applies exactly without changing anything. In Section 2.3 the bound (2.33) is left unchanged with V replaced by V^a. The lower bound (2.35) is replaced by

$$\langle V^a_{\Gamma_0} \rangle_{\psi_{\partial\Gamma_0}, V^a_{\Lambda-\Gamma_0}} \leqslant \frac{1}{8a + 8C_1 \ln L(i, \partial\Gamma_0)} \int_{-a}^{a} V^a(h)\, dh \tag{D2}$$

In order to show this upper bound, one has to show that F is decreasing. The method we used can be straightforwardly modified and Eq. (2.43) can be written in this case as

$$\begin{aligned}
-\frac{d}{dx} G(x) = & \sum_{\Delta \in \partial\Gamma_0, j \in \Delta} \Big\langle \delta(h_i - x)[\delta(h_j - a) - \delta(h_j + a)] \\
& \times \prod_{\substack{j' \in \Delta \\ j' \neq j}} \chi(h_{j'} \notin [-a, a]) \prod_{\substack{\Delta' \in \partial\Gamma_0 \\ \Delta' \neq \Delta}} \psi_{\Delta'} e^{-V^a_{\Lambda-\Gamma_0}} \Big\rangle \\
& + \frac{1}{2} \sum_{j \in \Lambda - \Gamma_0} \Big\langle [\delta(h_i - x)] \left[\frac{d}{dh_i} V^a(h_i) - \frac{d}{dh_i} V^a(-h_i) \right] \\
& \times \prod_{\Delta \in \Gamma_0} \psi_\Delta e^{-V^a_{\Lambda-\Gamma}} \Big\rangle
\end{aligned} \tag{D3}$$

Once again every element of this sum is positive. It remains to show the last part of the lemma. We have

$$\langle V^a(h_i) \rangle_{\psi_{\partial\Gamma_0}, V^a_{\Lambda-\Gamma_0}} = \int_{-a}^{+a} V^a(x)\, F(x)\, dx \tag{D4}$$

From the GKS inequality applied to functions of one variable we have

$$\frac{1}{2a} \int_{-a}^{a} (-V^a)(x)\, F(x)\, dx \geqslant \left[\frac{1}{2a} \int_{-a}^{a} (-V^a)(x)\, dx \right] \left[\frac{1}{2a} \int_{-a}^{a} F(x)\, dx \right] \tag{D5}$$

As a result,

$$\int_{-a}^{a} V^a(x)\, F(x)\, dx \leqslant \left[\int_{-a}^{a} V^a(x)\, dx \right] \left[\frac{1}{2a} \int_{-a}^{a} F(x)\, dx \right] \tag{D6}$$

Equation (2.50) still applies and we get the announced upper bound.

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